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From frequency-dependent mass and stiffness matrices to the dynamic response of elastic systems

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Abstract

More than three decades ago, Przemieniecki introduced a formulation for the free vibration analysis of bar and beam elements based on a power series of frequencies. In the present paper, the authors generalize this formulation for the analysis of the dynamic response of elastic systems submitted to arbitrary nodal loads as well as initial displacements. Based on the mode-superposition method, a set of coupled, higher-order differential equations of motion is transformed into a set of uncoupled second-order differential equations, which may be integrated by means of standard procedures. Motivation for this theoretical achievement is the hybrid boundary element method, which has been developed by the authors for time-dependent as well as frequency-dependent problems. This formulation, as a generalization of Pian's previous achievements for finite elements, yields a stiffness matrix for which only boundary integrals are required, for arbitrary domain shapes and any number of degrees of freedom. The use of higher-order frequency terms drastically improves numerical accuracy. The introduced modal assessment of the dynamic problem is applicable to any kind of finite element for which a generalized stiffness matrix is available. Some academic examples illustrate the theory. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Frequency-dependent matrices; Transient analysis; Boundary element methods

1. Introduction

1.1. Problem formulation

The time effect to be considered in this theoretical outline is due to the inertia of an elastic body. For the sake of simplicity, damping is not considered, since this effect, although not easy to be grasped physically, can generally be incorporated into an existing structural model by means of well established mathematical tools. Also, one is restricted to both material and geometric linear analysis. Particularization to problems of potential is straightforward.

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One is attempting to find the displacement field u_i with a corresponding stress field σ_{ij} that satisfies the dynamic partial differential equation

$$\sigma_{ij,j} + \overline{f}_i - \rho \ddot{u}_i = 0 \quad \text{in} \ \ \Omega \tag{1}$$

in a domain Ω, for given body forces $\overline{f}_i$ and specific body mass density ρ. Subscripts i and j may assume values 1, 2 and 3, as referred to global coordinates x, y and z, respectively. A subscript after a colon denotes derivative with respect to the corresponding coordinate direction. Repeated indices indicate a three-terms summation, in the general case of three-dimensional problems. A dot indicates derivative with respect to time.

The displacements must satisfy the boundary condition

$$u_i = \overline{u}_i \quad \text{along} \ \Gamma_u \tag{2}$$

for prescribed boundary displacements $\overline{u}_i$. Moreover, the stress field must be in equilibrium with prescribed forces $\overline{t}_i$ along the complementary part Γ_σ of the boundary

$$\sigma_{ij}\eta_j = \overline{t}_i \quad \text{along} \ \Gamma_\sigma. \tag{3}$$

All variables are functions of time. The elastic body is being observed during the time interval (t_0, t_1). Then, initial displacements as well as velocities in the domain have to be known at the time $t = t_0$, in general

$$\begin{matrix} u_i(t = t_0) = \overline{u}_i(t = t_0) \\ \dot{u}_i(t = t_0) = \overline{\dot{u}}_i(t = t_0) \end{matrix} \quad \text{in} \ \Omega. \tag{4}$$

A solution exactly satisfying all four equations above is possible only in certain particular cases. It is far beyond the scope of this paper to discuss the numeric models that can be developed to approximately deal with the proposed problem. However, a few words are necessary to introduce the motivation that is presently driving the authors.

In terms of spatial discretization, all mathematical approaches envisaged for solving the problem fall into either a domain or a boundary model, considering spatial allocation of the primary unknown parameters as well as whether domain or boundary integrals are to be carried out ("boundary element" formulations using internal nodal points as well as needing domain integration do also exist (Brebbia et al., 1984)). In the frame of a mathematical discretization model, Eq. (1) is usually transformed into a set of nodal equilibrium equations, which are exactly satisfied at the spatially distributed nodal points. The formulation is not based on a priori fulfillment of Eq. (1), although convergence of results is assured by increasing the number of nodal points. However, this is true only if the nodal points are distributed all over the domain (as in the case of both finite element and finite difference methods). In a plain boundary formulation, increasing the number of boundary nodal parameters does not improve results, at least beyond some threshold, unless fulfillment of Eq. (1) is taken as a premise (Dumont and de Oliveira, 1993a,b; de Oliveira, 1994). The developments of this paper apply to both approach types, although it is more relevant to a boundary discretization model, since focus is centered on the most accurate fulfillment of Eq. (1).

Moreover, time-dependent problems may be dealt with either in a time-domain formulation or in a frequency-domain frame. In a time-domain formulation, attempt is made to solve the spatial problem (as described in the above paragraph) for a given instant of time and then integrate the resulting nodal differential equations of time either numerically (generally) or analytically starting with the initial conditions given by Eq. (4). Preferentially, however, frequency-domain implementations are carried out. In this formulation, the field quantities as well as nodal variables are expressed as a series of products of separate space and time functions. The separation constants that arise in this formulation are usually identified with circular frequencies. As a consequence of this procedure, plain frequency analyses of a problem may be carried out, independently of the time variable (stationary problems), and, moreover, free-vibration characteristics of the elastic body may be investigated.

Quite frequently, however, one starts with the frequency-domain formulation of a problem and then combines the solution with the separated time-dependent functions in order to obtain the dynamic response of the elastic body to a given time-varying solicitation (transient analysis). This is the subject of the present paper.

1.2. An assessment of a formulation with higher-order frequency terms

Before entering the core of this paper, one should assess the problem of a frequency-domain formulation with higher-order frequency terms. For this sake, consider a truss element of constant cross-section A, length ℓ, Young modulus E and specific mass density ρ submitted to harmonic vibration. Writing $k = \omega/c$, in which ω is a given circular vibration frequency and $c = \sqrt{E/\rho}$ is the velocity of propagation of the longitudinal elastic wave, one expresses the corresponding effective, frequency-dependent stiffness matrix $\mathbf{K}$ of the two-degrees of freedom truss element, together with its eigenvalues ω_j, as

$$\mathbf{K} = EA\frac{k}{\sin(k\ell)}\begin{bmatrix}\cos(k\ell) & -1\\ -1 & \cos(k\ell)\end{bmatrix} \to \omega_0 = 0 \text{ (static case)}, \qquad \omega_j = \frac{j\pi c}{\ell} \quad j = 1,2,3,\ldots \tag{5}$$

These results are analytically exact. On the other hand, the matrix $\mathbf{K}$ in Eq. (5) may be expanded as a frequency series, as given below, up to the sixth power, together with its first non-zero vibration frequency

$$\mathbf{K} \approx \frac{EA}{\ell}\left(\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \frac{(k\ell)^2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} - \frac{(k\ell)^4}{45}\begin{bmatrix}8 & 7\\ 7 & 8\end{bmatrix} - \frac{(k\ell)^6}{15120}\begin{bmatrix}32 & 31\\ 31 & 32\end{bmatrix}\right) \to \omega_1 = 1.00438\frac{\pi c}{\ell}. \tag{6}$$

This eigenvalue is only 0.4% above the exact solution, according to Eq. (5). However, instead of such an accurate matrix, as given in Eq. (6), most analysts use its first approximation, which yields a 10% wrong vibration frequency

$$\mathbf{K} \approx \frac{EA}{\ell}\left(\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \frac{(k\ell)^2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) \to \omega_1 = 1.1026\frac{\pi c}{\ell}. \tag{7}$$

The use of higher-orders terms in stiffness and mass matrices, as outlined above, was apparently introduced by Przemieniecki (1968), but he has only developed truss and beam element matrices and only performed vibration analyses.

1.3. How general matrices with higher-order frequency terms originate

Moreover, it is worth mentioning that frequency-dependent matrices also may arise in the frame of a traditional finite element formulation, if one applies dynamic reduction of the degrees of freedom. Consider, for instance, a fixed-free bar (as illustrated in Fig. 2) made of two truss elements with matrices given by Eq. (7). Accounting for the boundary condition, both structure's matrix and first vibration frequency are

$$\mathbf{K} = \frac{EA}{\ell}\left(\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \frac{(k\ell)^2}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right) \to \omega_1 = 0.25646\frac{\pi c}{\ell}. \tag{8}$$

(This frequency is 2.6% higher than the exact one.) If one condenses statically the first, internal degree of freedom (Guyian's reduction, according to Przemieniecki (1968)), one obtains the reduced matrix and the corresponding free vibration frequency

$$\mathbf{K}_S = \frac{EA}{\ell}\left(\left[\frac{1}{2}\right] - (k\ell)^2\left[\frac{5}{12}\right]\right) \to \omega_1 = 0.34869\frac{\pi c}{\ell} \tag{9}$$

which is 35.9% higher than the target value given in Eq. (8).

On the other hand, one may condense dynamically the first degree of freedom of the matrix given in Eq. (8) by expanding the frequency-dependent inverse matrix in series of $(k\ell)$. Keeping $(k\ell)^6$ as the highest power in the resulting condensed matrix $\mathbf{K}_\mathrm{D} = \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}$, one obtains

$$\mathbf{K}_\mathrm{D} = \frac{EA}{\ell}\left(\left[\frac{1}{2}\right] - (k\ell)^2\left[\frac{2}{3}\right] - (k\ell)^4\left[\frac{1}{8}\right] - (k\ell)^6\left[\frac{1}{24}\right]\right) \to \omega_1 = 0.25717\frac{\pi c}{\ell}. \tag{10}$$

This frequency value is only 0.27% higher than the target one given in Eq. (8).

It is straightforward to develop in series the inverse of a general frequency-dependent matrix. The algorithm (written for flexibility matrices $\mathbf{F}$ instead of stiffness matrices $\mathbf{K}$) is given in Eq. (38), as a very particular case of the inverse of a matrix power series in which the first matrix coefficient is singular.

This simple example illustrates how matrices with higher-order frequency terms may originate from conventional formulations. Dynamic condensation is not the subject of this paper. However, it is worth observing that a condensation method that introduces higher-order terms might be developed in combination with some more traditional ones, as the component mode synthesis (Petyt, 1990; Hou, 1969), or the procedure proposed by Paz (1997).

In Section 2 of this paper, one is aimed at showing that general frequency-dependent effective stiffness matrices may be obtained, in the frame of the hybrid boundary element method, for seemingly any kind of problem. However, the reader may skip this section and go directly to the core of the present subject, starting in Section 3, which presupposes that a general finite element stiffness matrix with higher-order frequency terms is already available.

2. The hybrid boundary element method in the frequency domain

In the present hybrid formulation (Dumont and de Oliveira, 1997; 1998; 1999a,b), one assumes that a displacement field

$$\tilde{u}_i = u_{im}d_m(t) \quad \text{along } \Gamma \qquad \text{such that } \tilde{u}_i = \overline{u}_i \quad \text{along } \Gamma_u \tag{11}$$

is known along the boundary in terms of polynomial interpolation functions u_{im} and some time-dependent nodal displacement parameters $d_m(t)$, in which the subscript m refers to each one of the degrees of freedom of the discretized model.

One also assumes a different displacement field

$$u_i = u_i^* + u_i^\mathrm{p} \quad \text{in } \Omega \tag{12}$$

for the entire domain, in such a way that the dynamic equilibrium Eq. (1) is identically satisfied. It means that one can define an arbitrary particular solution u_i^p, such that the corresponding stress field σ_{ij}^p satisfies the equation

$$\sigma_{ij,j}^\mathrm{p} + \overline{f}_i - \rho\ddot{u}_i^\mathrm{p} = 0 \quad \text{in } \Omega \tag{13}$$

and, most important, it means that one can find a homogeneous solution u_i^* with corresponding stress field σ_{ij}^* that satisfies identically

$$\sigma_{ij,j}^* - \rho\ddot{u}_i^* = 0 \qquad \text{in } \Omega \tag{14}$$

This characterizes a fundamental solution

$$u_i^* = u_{im}^*(t)p_m^*(t) \quad \text{and} \quad \sigma_{ij}^* = \sigma_{ijm}^*(t)p_m^*(t) \tag{15}$$

to be obtained in terms of some time-dependent nodal force parameters $p_m^*(t)$, in which the subscript m refers to each one of the degrees of freedom of the discretized model.

Given the assumptions above, one shall look for a means of relating the fields $\tilde{u}_i$, defined on Γ by Eq. (11), and u_i^*, defined in Ω by Eqs. (12)–(15), in such a way that Eq. (3) is best satisfied. This may be achieved by means of a variational principle, in terms of the Hellinger–Reissner potential, generalized for time-dependent problems (Dumont and de Oliveira, 1997),

$$\int_{t_0}^{t_1}\left(-\int_\Omega(\delta\sigma_{ij,j}-\rho\delta\ddot{u}_i)(u_i-\tilde{u}_i)\,\mathrm{d}\Omega+\int_\Gamma\delta\sigma_{ij}\eta_j(u_i-\tilde{u}_i)\,\mathrm{d}\Gamma+\int_\Omega\delta\tilde{u}_i(\sigma_{ij,j}+\overline{f}_i-\rho\ddot{u}_i)\,\mathrm{d}\Omega\right.$$
$$\left.-\int_\Gamma\delta\tilde{u}_i(\sigma_{ij}\eta_j-\overline{t}_i)\,\mathrm{d}\Gamma\right)\mathrm{d}t=0 \tag{16}$$

in which one assumes as necessary variational prerequisites that $\delta\tilde{u}_i=0$ along Γ_u and that $\delta u_i=0$ at both time interval extremities t_0 and t_1. In the above equation, η_j are the direction cosines of the outward normal to the surface element $\mathrm{d}\Gamma$.

After interpolation of the variables $\tilde{u}_i$, according to Eq. (11), as well as u_i^* and σ_{ij}^*, according to Eq. (15), one arrives at the expression, for a given time instant t

$$\delta\mathbf{p}^{*\mathrm{T}}(\mathbf{F}\mathbf{p}^*-\mathbf{H}\mathbf{d}+\mathbf{b})-\delta\mathbf{d}^{\mathrm{T}}\left(\mathbf{H}^{\mathrm{T}}\mathbf{p}^*+\mathbf{t}-\mathbf{p}\right)=\mathbf{0}. \tag{17}$$

This equation is expressed in matrix notation, for convenience. The quantities $\mathbf{p}^*$ and $\mathbf{d}$ are vectors containing the nodal parameters p_m^* and d_m, respectively – the primary unknowns of the problem. The symmetric flexibility matrix $\mathbf{F}$, the cinematic transformation matrix $\mathbf{H}$ and the vector $\mathbf{b}$ of nodal displacements equivalent to body forces are defined in terms of boundary integrals as

$$\begin{bmatrix}\mathbf{F}\\ \mathbf{H}^{\mathrm{T}}\\ \mathbf{b}^{\mathrm{T}}\end{bmatrix}\equiv\begin{bmatrix}F_{mn}\\ H_{mn}\\ b_n\end{bmatrix}=\int_\Gamma\begin{Bmatrix}u_{im}^*\\ u_{im}\\ u_i^{\mathrm{p}}\end{Bmatrix}\langle\sigma_{ijn}^*\eta_j\rangle\,\mathrm{d}\Gamma+\begin{Bmatrix}u_{im}^*\\ u_{im}\\ u_i^{\mathrm{p}}\end{Bmatrix}\langle\delta_{in}\rangle. \tag{18}$$

Owing to the singularity of the fundamental solution, the boundary integral represented above is singular and has to be split into a Cauchy principal value and a discontinuous term. Related to this singularity, a generalized Kronecker delta is introduced, meaning that $\delta_{in}=0$ in general, except if the indices i and n refer to the same degree of freedom, when $\delta_{in}=1$.

Nodal forces vectors $\mathbf{t}$ and $\mathbf{p}$, equivalent to body forces $\overline{f}_i$ and traction forces $\overline{t}_i$, respectively, as introduced in Eq. (17), are defined as

$$[\,\mathbf{t}\quad\mathbf{p}\,]\equiv[\,t_m\quad p_m\,]=\int_\Gamma\{u_{im}\}\langle\sigma_{ij}^{\mathrm{p}}\eta_j\quad\overline{t}_i\rangle\,\mathrm{d}\Gamma. \tag{19}$$

For arbitrary values of $\delta\mathbf{p}^*$ and $\delta\mathbf{d}$, Eq. (17) becomes

$$\mathbf{F}\mathbf{p}^*-\mathbf{H}\mathbf{d}+\mathbf{b}=\mathbf{0}\quad\text{and}\quad\mathbf{H}^{\mathrm{T}}\mathbf{p}^*+\mathbf{t}-\mathbf{p}=\mathbf{0}, \tag{20}$$

or

$$\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{H}\mathbf{d}=\mathbf{p}-\mathbf{t}-\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{b} \tag{21}$$

in which

$$\mathbf{H}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{H}=\mathbf{K} \tag{22}$$

constitutes a symmetric, positive semi-definite stiffness matrix that transforms nodal displacements $\mathbf{d}$ into nodal forces in equilibrium with the set of equivalent nodal forces defined at the right-hand side of Eq. (21).

In all matrices defined above, time-independent terms may be considered explicitly, as they correspond to the static formulation. For matrices **F** and **H**, in particular

$$\mathbf{F} = \mathbf{F_0} + \mathbf{F}(t), \qquad \mathbf{H} = \mathbf{H_0} + \mathbf{H}(t). \tag{23}$$

The evaluation of $\mathbf{F}(t)$ by means of the integral indicated in Eq. (18) offers no mathematical difficulty. On the other hand, elements about the main diagonal of the static flexibility matrix $\mathbf{F_0}$, for m and n referring to the same node, cannot be evaluated by means of this integral. This mathematical impossibility is consistent with the assumption that the nodal point is situated outside the domain Ω, although infinitely close to it. The determination of these elements has to be carried out indirectly by requiring that $\mathbf{F_0}$ satisfies the orthogonality criterion (Dumont, 1987, 1989; Dumont and de Oliveira, 1995)

$$\mathbf{F_0}\mathbf{V} = \mathbf{0} \tag{24}$$

in which **V** is a basis of eigenvectors corresponding to zero-eigenvalues of the matrix $\mathbf{H}_0^{\mathrm{T}}$

$$\mathbf{H}_0^{\mathrm{T}}\mathbf{V} = \mathbf{0}. \tag{25}$$

Eqs. (17)–(22), as established above for time-dependent problems, are formally the same ones obtained by the first author for static problems (Dumont, 1987,1989). The numerical implementation of this formulation for a time-dependent fundamental solution is very complicated, since an additional singularity with respect to time has to be considered. However, time-harmonic as well as general transient problems may be considered in the frame of a frequency-domain formulation.

2.1. Frequency-domain formulation

2.1.1. Basic equations

For this sake, one first assumes that the trial solutions $\tilde{u}_i$ along Γ, Eq. (11), and u_i^* in Ω, Eq. (15), may be expressed in terms of separate variables of space and time, for a given circular frequency of vibration ω

$$\tilde{u}_i = u_{im} d_m(\omega)\tau(t,\omega) \quad \text{along } \Gamma, \tag{26}$$

$$u_i^* = u_{im}^*(\omega)p_m^*(\omega)\tau(t,\omega), \qquad \sigma_{ij}^* = \sigma_{ijm}^*(\omega)p_m^*(\omega)\tau(t,\omega) \quad \text{in } \Omega, \tag{27}$$

where $\tau(t,\omega)$ is defined in such a way that

$$\frac{\partial^2 \tau(t,\omega)}{\partial t^2} = -\omega^2 \tau(t,\omega). \tag{28}$$

The formulation above relies on the existence of a fundamental solution, as introduced in Eq. (27), which, by definition, satisfies Eq. (14) for a given circular frequency ω

$$\sigma_{ij,j}^*(\omega) + \omega^2 \rho u_i^*(\omega) = 0 \quad \text{in } \Omega. \tag{29}$$

Such fundamental solutions are well known in the literature for both elasticity and potential problems. For two-dimensional problems, they are linear combinations of Bessel functions of order zero of the first and second kinds. An imposition, known as Sommerfield radiation condition, that at infinity the velocity $\dot{u}_i^*(t)$ must tend to zero apparently forces that the fundamental solution be expressed as a complex function. However, it is the authors' point of view that such complex fundamental solution is needed only in the frame of the conventional boundary element method, which does not consider the balance of forces adequately (Dumont, 1998). The present hybrid formulation is variationally consistent, which ensures that equilibrated actions at a finite region dissipate at infinity. Moreover, one notes that Bessel functions together with their derivatives tend to zero as the argument increases, thus assuring automatic satisfaction of the Sommerfield radiation condition at infinity. (This issue certainly deserves a separate paper for a proper discussion.) One might, for the sake of conciseness, express $\tau(t,\omega)$ as complex. In this case, the nodal quantities $\mathbf{d}(\omega)$ and $\mathbf{p}^*(\omega)$ in Eqs. (26) and (27) are also complex. However, the products $\mathbf{d}(\omega)\tau(t,\omega)$ and $\mathbf{p}^*(\omega)\tau(t,\omega)$ must be always real. All implementations of the present formulation, for problems in both finite and infinite regions, consider a fundamental solution expressed in terms of real variables, which simplifies the code and reduces computational time at no expense of accuracy.

The solution of Eq. (29) may be adequately expressed as

$$u^*_{im}(\omega) \leftarrow u^*_{im}(0) + u^*_{im}(\omega) \quad \text{and} \quad \sigma^*_{ijm}(\omega) \leftarrow \sigma^*_{ijm}(0) + \sigma^*_{ijm}(\omega) \tag{30}$$

in which $u^*_{im}(0)$ and $\sigma^*_{ijm}(0)$ correspond to the static fundamental solution. On the other hand, the frequency-dependent terms $u^*_{im}(\omega)$ and $\sigma^*_{ijm}(\omega)$ are by construction non-singular functions that require no special consideration for the sake of the integration indicated in Eq. (18). For two-dimensional isotropic problems of potential, for instance, the solution of the Helmholtz equation in terms of a radial distance r and the frequency number k is the two-terms expression

$$\theta^* = \frac{-1}{2\pi}\ln(r) + \frac{-1}{2\pi}\left(\frac{\pi}{2}\mathrm{Bessel}Y(0,kr) - \ln(r) - \left(\ln\left(\frac{k}{2}\right) + \gamma\right)\mathrm{Bessel}J(0,kr)\right). \tag{31}$$

Then, according to Eqs. (26)–(28), Eq. (20) become, for a given circular frequency ω

$$\begin{aligned} &(\mathbf{F}(\omega)\mathbf{p}^*(\omega) - \mathbf{H}(\omega)\mathbf{d}(\omega) + \mathbf{b}(\omega))\tau(t,\omega) = \mathbf{0}, \\ &\left(\mathbf{H}^{\mathbf{T}}(\omega)\mathbf{p}^*(\omega) + \mathbf{t}(\omega) - \mathbf{p}(\omega)\right)\tau(t,\omega) = \mathbf{0} \end{aligned} \tag{32}$$

in which $\mathbf{b}(\omega), \mathbf{t}(\omega)$ and $\mathbf{p}(\omega)$ are, according to Eqs. (18) and (19), the harmonic components of the general time-dependent vectors $\mathbf{b}$, $\mathbf{t}$ and $\mathbf{p}$, respectively.

As a consequence of writing the fundamental solution in the shape of Eq. (30), the matrices $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{b}$ above may be formally represented as

$$\mathbf{F}(\omega) \equiv \mathbf{F}_0 + \mathbf{F}_\omega, \quad \mathbf{H}(\omega) \equiv \mathbf{H}_0 + \mathbf{H}_\omega, \quad \mathbf{b}(\omega) \equiv \mathbf{b}_0 + \mathbf{b}_\omega. \tag{33}$$

In these equations, the frequency-dependent terms $\mathbf{F}_\omega, \mathbf{H}_\omega, \mathbf{b}_\omega$ involve no singularities. The terms $\mathbf{F}_0, \mathbf{H}_0, \mathbf{b}_0$ correspond to the matrices of a static formulation, with integration singularities that can be dealt with adequately. Moreover, the terms about the main diagonal of $\mathbf{F}_0$ can be obtained by means of spectral properties, as given by Eq. (24), that affect exclusively the static formulation, for both cases of finite and infinite regions.

For a periodic loading applied with a certain circular frequency ω, the stationary response of the structure is obtained from Eq. (32) as

$$\begin{aligned} &\mathbf{F}(\omega)\mathbf{p}^*(\omega) - \mathbf{H}(\omega)\mathbf{d}(\omega) + \mathbf{b}(\omega) = \mathbf{0}, \\ &\mathbf{H}^{\mathbf{T}}(\omega)\mathbf{p}^*(\omega) + \mathbf{t}(\omega) - \mathbf{p}(\omega) = \mathbf{0}. \end{aligned} \tag{34}$$

2.1.2. Transient analysis in the frequency domain

Instead of formulating a problem for a given value of the circular frequency, one may express the fundamental solution, Eq. (30), as a power series of frequencies. In such a case, series expansion of the fundamental solution of Eq. (31), for instance, yields

$$\theta^* = -\frac{\ln(r)}{2\pi} + \frac{k^2r^2}{27648\pi}\left[(\ln(r) - 1)3456 - (216\ \ln(r) - 324)k^2r^2 + (6\ \ln(r) - 11)k^4r^4\right] + \mathrm{O}(r^8). \tag{35}$$

As a consequence, the matrices **F** and **H**, defined in Eq. (18), as well as **K**, defined in Eq. (22), also become power series of frequencies with an arbitrary number n of terms

$$\mathbf{F} = \sum_{i=0}^{n} \omega^{2i}\mathbf{F}_i, \quad \mathbf{H} = \sum_{i=0}^{n} \omega^{2i}\mathbf{H}_i, \quad \mathbf{K} = \sum_{i=0}^{n} \omega^{2i}\mathbf{K}_i. \tag{36}$$

In the above equation and in the following equations, the indices i, j and n are re-introduced with different meanings, as compared with the initial sections of this paper. Moreover, summation is only assumed when explicitly indicated.

2.1.3. On the evaluation of the stiffness matrix K as a power series of frequencies

In order to express the stiffness matrix **K** as a power series of frequencies, according to Eq. (36), one first has to invert the flexibility matrix **F**, as expressed in Eq. (21).

For an infinite domain, the frequency-free term $\mathbf{F}_0$ corresponding to the static formulation, as expanded in Eq. (33), is non-singular. In this case, it is straightforward to demonstrate that the power series matrix **X** with n terms

$$\mathbf{X} = \sum_{i=0}^{n} \omega^{2i}\mathbf{X}_i \tag{37}$$

in which

$$\mathbf{X}_0 = \mathbf{F}_0^{-1} \text{ and } \mathbf{X}_i = -\mathbf{X}_0 \sum_{j=1}^{i} \mathbf{F}_j \mathbf{X}_{i-j}, \quad i = 1 \ldots n \tag{38}$$

is the unique inverse of **F**, such that

$$\mathbf{XF} = \mathbf{FX} = \mathbf{I} + \mathrm{O}(\omega^{2n+2}). \tag{39}$$

(Eqs. (37)–(39) apply analogously for the inversion of a frequency-dependent stiffness matrix, as proposed in the introduction.)

In case of a finite domain, however, the frequency-free term $\mathbf{F}_0$ corresponding to the static formulation is singular, as indicated in Eq. (24). Then, the evaluation of a unique inverse matrix **X** that satisfies Eq. (39) is no longer straightforward, since it requires advanced use of the theory of generalized inverse matrices (Ben-Israel and Greville, 1980; Zielke, 1970; Schulz, 1933). A detailed paper on this subject is being prepared for publication. Its main results are summarized below.

For a finite domain, the inverse matrix **X** has to be expressed as

$$\mathbf{X} = \sum_{i=-1}^{n} \omega^{2i}\mathbf{X}_i \tag{40}$$

with an additional term $\omega^{-2}\mathbf{X}_{-1}$, as compared with Eq. (37), in which

$$\mathbf{X}_{-1} = \mathbf{V}\left(\mathbf{V}^{\mathrm{T}}\mathbf{F}_1\mathbf{V}\right)^{-1}\mathbf{V}^{\mathrm{T}}. \tag{41}$$

The matrix $\mathbf{F}_1$ is non-singular, since it is physically related to the first mass matrix of the elastic body. Introducing an auxiliary matrix **Y**

$$\mathbf{Y} = \mathbf{F}_1^{-1}\left(\mathbf{I} - \mathbf{V}\mathbf{V}^{\mathrm{T}}\right) \tag{42}$$

one may demonstrate that the matrix coefficient $\mathbf{X}_0$ of Eq. (40) is expressed as

$$\mathbf{X}_0 = \mathbf{Y}\left(\mathbf{Y}^{\mathrm{T}}\mathbf{F}_0\mathbf{Y} + \mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{Y}^{\mathrm{T}} - \mathbf{X}_{-1}\mathbf{F}_2\mathbf{X}_{-1}. \tag{43}$$

The remaining terms are obtained recurrently as

$$\mathbf{X}_i = -\mathbf{X}_0\sum_{j=1}^{i+1}\mathbf{F}_j\mathbf{X}_{i-j} - \mathbf{X}_{-1}\sum_{j=1}^{i+1}\mathbf{F}_{j+1}\mathbf{X}_{i-j}, \quad i = 1\ldots n. \tag{44}$$

Although there is a negative power of frequency in Eq. (40) the matrix product

$$\mathbf{S} \equiv \mathbf{F}^{-1}\mathbf{H} = \sum_{i=0}^{n}\omega^{2i}\sum_{j=-1}^{i}\mathbf{X}_j\mathbf{H}_{i-j} \equiv \sum_{i=0}^{n}\omega^{2i}\mathbf{S}_i \tag{45}$$

required both for the expression of $\mathbf{p}^*$ in Eq. (20), as a function of $\mathbf{d}$, and as an intermediate step in the evaluation of the stiffness matrix $\mathbf{K}$, does not contain the term $\omega^{-2}\mathbf{X}_{-1}$, since

$$\mathbf{H}_0^{\mathbf{T}}\mathbf{X}_{-1} = \mathbf{0} \tag{46}$$

according to Eqs. (25) and (41).

Finally, one expresses the stiffness matrix $\mathbf{K}$ as the power series

$$\mathbf{K} \equiv \mathbf{H}^{\mathbf{T}}\mathbf{F}^{-1}\mathbf{H} = \sum_{i=0}^{n}\omega^{2i}\sum_{j=0}^{i}\mathbf{H}_j^{\mathbf{T}}\mathbf{S}_j. \tag{47}$$

3. From frequency to time-domain formulation

According to the definition of the time function $\tau(t,\omega)$ in Eq. (28), one may compose the time-dependent vector $\mathbf{d}$ of nodal displacements one is looking for as the Fourier series

$$\mathbf{d} \equiv \mathbf{d}(t) = \sum_{j=-\infty}^{\infty}\mathbf{d}_j\tau(t,\omega_j) \tag{48}$$

in which $\omega_j = 2\pi j/T$ is a circular frequency defined in terms of the integer number j and the time interval $T = t_1 - t_0$. In general, this is a truncated series. Moreover, it may be expressed either in the compact expression presented above, for which a complex representation is required, or as sinus and co-sinus series for non-negative values of j. The specific aspect of the Fourier series introduced above is not relevant in this paper.

In a frequency formulation, given a vector of equivalent time-dependent forces $\mathbf{p}(t)$ acting on an elastic body, the behavior of the damping-free structure may be modeled as

$$\sum_{j=-\infty}^{\infty}\left(\mathbf{K}_0\mathbf{d}_j\tau(t,\omega_j) - \sum_{i=1}^{n}\omega_j^{2i}\mathbf{M}_i\mathbf{d}_j\tau(t,\omega_j)\right) = \mathbf{p}(t). \tag{49}$$

In this equation, one expresses $\mathbf{K}_0$ explicitly as the stiffness matrix of the static discrete-element formulation and renames the remaining terms of the power series of $\mathbf{K}$ in Eq. (36) as $-\mathbf{M}_i$, as generalized mass matrices, although they constitute a blending of mass and stiffness matrices, in the generalized frequency-dependent expression (Przemieniecki, 1968). The only exception is the matrix $\mathbf{M}_1$, which corresponds to the mass matrix obtained in the conventional formulation that truncates after ω^2. The vectors $\mathbf{d}_j$ of displacements are the unknowns of the problem, to be determined as functions of the vector $\mathbf{p}(t)$ of applied equivalent nodal forces as well as of the initial nodal displacements and velocities. The number n of frequency-related matrices is arbitrary. The advantage of such a formulation based on a power series of frequencies is that it provides a more accurate fulfillment of the dynamic differential equilibrium equations of stresses at internal points of the elastic body (Dumont and de Oliveira, 1997).

According to Eq. (48), one may express Eq. (49) alternatively as

$$\mathbf{K}_0\mathbf{d} - \sum_{i=1}^{n}(-1)^i\mathbf{M}_i\frac{\partial^{2i}\mathbf{d}}{\partial t^{2i}} = \mathbf{p}(t) \tag{50}$$

which is a coupled set of higher-order time derivatives that makes use of the matrices obtained in the frequency formulation.

4. An assessment of the non-linear eigenvalue problem related to Eq. (49)

Before further manipulating Eq. (50), it is necessary to solve the eigenvalue problem related to Eq. (49)

$$\mathbf{K}_0\Phi - \sum_{i=1}^{n}\mathbf{M}_i\Phi\Omega^{2i} = \mathbf{0} \tag{51}$$

in which Ω^2 is a diagonal matrix with as many eigenvalues ω^2 as the number of degrees of freedom of the structure and Φ is a matrix whose columns are the corresponding eigenvectors. This non-linear eigenvalue problem is difficult to deal with, since numerical convergence cannot be easily assured and round-off errors occur unavoidably. Assuming that this eigenvalue problem has been solved adequately, one observes that its solution is part of the solution of the augmented eigenvalue problem

$$\left(\begin{bmatrix}\mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n\\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \dots & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \dots & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Phi_{00} & \Phi_{01} & \dots & \Phi_{0,n-1}\\ \Phi_{10} & \Phi_{11} & \dots & \Phi_{1,n-1}\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_{n-1,0} & \Phi_{n-1,1} & \dots & \Phi_{n-1,n-1}\end{bmatrix} - \begin{bmatrix}\mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n\\ \mathbf{M}_2 & \mathbf{M}_3 & \dots & \dots & \mathbf{0}\\ \mathbf{M}_3 & \vdots & \ddots & \dots & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Phi_{00} & \Phi_{01} & \dots & \Phi_{0,n-1}\\ \Phi_{10} & \Phi_{11} & \dots & \Phi_{1,n-1}\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_{n-1,0} & \Phi_{n-1,1} & \dots & \Phi_{n-1,n-1}\end{bmatrix}\begin{bmatrix}\Omega^2 & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \Omega_1^2 & \dots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \dots & \Omega_{n-1}^2\end{bmatrix}\right) = \mathbf{0} \tag{52}$$

in which

$$\Phi_{00} \equiv \Phi, \quad \Omega_0^2 \equiv \Omega^2 \quad \text{and} \quad \Phi_{ij} = \Phi_{0j}\Omega_j^{2i} \qquad i = 1,\dots,n-1, \quad j = 0,\dots,n-1. \tag{53}$$

The enlarged eigenvalues and eigenvectors of Eqs. (52) and (53) are in general complex. However, only evaluation of the real subsets Ω and Φ is required in a practical application.

Since the augmented eigenvalue problem expressed by Eq. (52) is linear in Ω_j^2, the corresponding augmented eigenvectors constitute an orthogonal, however still not orthonormal, basis. One may state as a normalization criterion for these eigenvectors that

$$\begin{bmatrix} \Phi_{00}^{\mathbf{T}} & \Phi_{10}^{\mathbf{T}} & \dots & \Phi_{n-1,0}^{\mathbf{T}} \\ \Phi_{01}^{\mathbf{T}} & \Phi_{11}^{\mathbf{T}} & \dots & \Phi_{n-1,1}^{\mathbf{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{0,n-1}^{\mathbf{T}} & \Phi_{1,n-1}^{\mathbf{T}} & \dots & \Phi_{n-1,n-1}^{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{M}_2 & \mathbf{M}_3 & \dots & \dots & \mathbf{0} \\ \mathbf{M}_3 & \vdots & \ddots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi_{00} & \Phi_{01} & \dots & \Phi_{0,n-1} \\ \Phi_{10} & \Phi_{11} & \dots & \Phi_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n-1,0} & \Phi_{n-1,1} & \dots & \Phi_{n-1,n-1} \end{bmatrix} = \mathbf{I}. \quad (54)$$

Evaluating the submatrix (0,0) of the system above and taking into account Eq. (53), it follows that $\Phi_{00} \equiv \Phi$ is an orthonormal basis only if

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\Omega^{2j-2i}\Phi^{\mathbf{T}}\mathbf{M}_j\Phi\Omega^{2i-2} = \mathbf{I}. \quad (55)$$

Let $\tilde{\Phi}$ be a set of non-normalized eigenvectors that satisfy both Eqs. (51) and (52). One may relate these eigenvectors to the normalized ones by means of a diagonal matrix Λ

$$\Phi = \tilde{\Phi}\Lambda. \quad (56)$$

Then, substituting this expression of Φ into Eq. (55) yields

$$\Lambda = \left(\sum_{i=1}^{n}\sum_{j=i}^{n}\Omega^{2j-2i}\tilde{\Phi}^{\mathbf{T}}\mathbf{M}_j\tilde{\Phi}\Omega^{2i-2}\right)^{-1/2}. \quad (57)$$

Moreover, one may state that, once Eq. (54) holds for normalized eigenvectors, premultiplication of Eq. (52) by the transpose of the augmented basis of eigenvectors yields an augmented set of uncoupled equations in Ω_j^2. Taking into account Eq. (55), the subset of equations related to the subscript 0 – the only one of interest – results in

$$\left(\Phi^{\mathbf{T}}\mathbf{K}_0\Phi + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\Omega^{2i}\Phi^{\mathbf{T}}\mathbf{M}_{j+i}\Phi\Omega^{2j}\right) = \Omega^2. \quad (58)$$

This equation, which relates frequencies to mass and stiffness matrices, is the most important achievement of this paper, since it enables a remarkable simplification of Eq. (50), as it shall be demonstrated presently.

5. Use of a mode-superposition procedure

Independently of any assumption on the shape of the time-dependent displacements vector $\mathbf{d}(t)$ in Eq. (50), one may introduce a set of auxiliary displacements vectors $\mathbf{d}_{(i)}(t)$, in which the subscripts in brackets indicate that they constitute a set (other than the one of Eqs. (48) and (49)), such that

$$\mathbf{d}_{(i)} = (-1)^i\frac{\partial^{2i}\mathbf{d}}{\partial t^{2i}} \quad i = 1,\dots,n-1. \quad (59)$$

According to that, Eq. (50) may also be rewritten as an augmented system

$$\begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{d}_{(1)} \\ \mathbf{d}_{(2)} \\ \vdots \\ \mathbf{d}_{(n-1)} \end{Bmatrix} + \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{M}_2 & \mathbf{M}_3 & \dots & \dots & \mathbf{0} \\ \mathbf{M}_3 & \vdots & \ddots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{d}}_{(1)} \\ \ddot{\mathbf{d}}_{(2)} \\ \vdots \\ \ddot{\mathbf{d}}_{(n-1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix} \tag{60}$$

in which two upper dots indicate the second derivative with respect to time.

Now, starting from Eq. (59), one approximates the time-dependent displacements $\mathbf{d}(t)$ and $\mathbf{d}_{(i)}(t)$ as a finite sum of contributions of the augmented (normalized) eigenvectors Φ_{i0} introduced as the first column of enlarged eigenvectors in Eq. (52), multiplied by a vector of amplitudes $\eta \equiv \eta(t)$, which are the new unknowns of the problem

$$\begin{Bmatrix} \mathbf{d} \\ \mathbf{d}_{(1)} \\ \vdots \\ \mathbf{d}_{(n-1)} \end{Bmatrix} = \begin{bmatrix} \Phi \\ \Phi\Omega^2 \\ \vdots \\ \Phi\Omega^{2n-2} \end{bmatrix} \eta. \tag{61}$$

According to that, Eq. (60) becomes

$$\begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi \\ \Phi\Omega^2 \\ \vdots \\ \Phi\Omega^{2n-2} \end{bmatrix} \eta + \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{M}_2 & \mathbf{M}_3 & \dots & \dots & \mathbf{0} \\ \mathbf{M}_3 & \vdots & \ddots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi \\ \Phi\Omega^2 \\ \vdots \\ \Phi\Omega^{2n-2} \end{bmatrix} \ddot{\eta} = \begin{Bmatrix} \mathbf{p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix}. \tag{62}$$

Finally, premultiplying this equation by $\Phi_{i0}^{\mathbf{T}}$, taking into account Eq. (53) and considering that the eigenvectors are normalized according to Eq. (55), such that Eq. (58) is satisfied, one arrives at a very simple expression for the submatrix $(0,0)$ of the augmented system of equations

$$\Omega^2\eta + \ddot{\eta} = \Phi^{\mathbf{T}}\mathbf{p}. \tag{63}$$

This equation constitutes an uncoupled system with as many second-order differential equations of time as the number of eigenvectors one may consider of interest in the complete set Φ. As a consequence, each separate element of η may be evaluated either numerically, via a finite-difference scheme, or analytically, by means of Duhamel's integral (Przemieniecki, 1968), for given initial values of nodal displacements and velocities.

6. Consideration of initial displacements and velocities

For non-homogeneous initial conditions, it is necessary to express $\eta(t = t_0)$ and $\dot{\eta}(t = t_0)$ as functions of the initial nodal displacements $\mathbf{d}(t = t_0)$ and velocities $\dot{\mathbf{d}}(t = t_0)$. For this sake, one has to solve the generally rectangular system of Eq. (61) in terms of the unknowns η. First of all, one premultiplies both sides of Eq. (61) by the augmented stiffness matrix of Eq. (60) and, subsequently, premultiplies the resulting equation by $\Omega_{i0}^{\mathbf{T}}$. Then, it results

$$\begin{bmatrix} \Phi^{\mathrm{T}} & \Omega^2\Phi^{\mathrm{T}} & \ldots & \Omega^{2n-2}\Phi^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \ldots & \mathbf{M}_n \\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{d}_{(1)} \\ \vdots \\ \mathbf{d}_{(n-1)} \end{bmatrix} = \Omega^2\eta \tag{64}$$

since the eigenvectors are orthonormal and Eq. (58) holds. Performing the matrix operations indicated in this equation and considering Eq. (59), one arrives at

$$\eta = \Omega^{-2}\Phi^{\mathrm{T}}\mathbf{K}_0\mathbf{d} + \sum_{i=1}^{n-1}\Omega^{2i-2}\Phi^{\mathrm{T}}\sum_{j=1}^{n-i}\mathbf{M}_{j+i}(-1)^j\frac{\partial^{2j}\mathbf{d}}{\partial t^{2j}}. \tag{65}$$

However, this equation is only applicable if $\mathbf{d}$ and all its $2n-1$ derivatives are known at the beginning of the time interval. Since in general only displacements and velocities are known, one is forced to obtain an alternative expression.

Substituting the values of $\mathbf{d}_{(i)}(t),\ i>0$, in Eq. (64), for their expressions given in Eq. (61), one obtains

$$\begin{bmatrix} \Phi^{\mathrm{T}} & \Omega^2\Phi^{\mathrm{T}} & \ldots & \Omega^{2n-2}\Phi^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \ldots & \mathbf{M}_n \\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \Phi\Omega^2\eta \\ \vdots \\ \Phi\Omega^{2n-2}\eta \end{bmatrix} = \Omega^2\eta. \tag{66}$$

Then, performing all matrix operations indicated in this equation, it results that

$$\left[\Phi^{\mathrm{T}}\mathbf{K}_0\Phi\right]\eta = \Phi^{\mathrm{T}}\mathbf{K}_0\mathbf{d} \tag{67}$$

since, according to Eq. (58), the series of matrix products multiplying η simplifies to the term in brackets. Finally, one concludes with no further assumption that, if Eq. (61) holds

$$\eta = \left[\Phi^{\mathrm{T}}\mathbf{K}_0\Phi\right]^{-1}\Phi^{\mathrm{T}}\mathbf{K}_0\mathbf{d} \tag{68}$$

also holds and, consequently,

$$\begin{aligned} \eta(t=t_0) &= \left[\Phi^{\mathrm{T}}\mathbf{K}_0\Phi\right]^{-1}\Phi^{\mathrm{T}}\mathbf{K}_0\mathbf{d}(t=t_0), \\ \dot{\eta}(t=t_0) &= \left[\Phi^{\mathrm{T}}\mathbf{K}_0\Phi\right]^{-1}\Phi^{\mathrm{T}}\mathbf{K}_0\dot{\mathbf{d}}(t=t_0). \end{aligned} \tag{69}$$

Note that one could have written the sequence of equations

$$\begin{aligned} \mathbf{d} &= \Phi\eta, \\ \mathbf{K}_0\mathbf{d} &= \mathbf{K}_0\Phi\eta, \\ \Phi^{\mathrm{T}}\mathbf{K}_0\mathbf{d} &= \Phi^{\mathrm{T}}\mathbf{K}_0\Phi\eta, \end{aligned} \tag{70}$$

thus arriving at Eq. (68) by means of a very simple procedure. However, premultiplying the second of equations above with matrix Φ^{T}, which may in general be rectangular (if only some deformation modes are of interest), needs a justification, which only occurs in the frame of the procedure just outlined: use of the orthogonal property expressed by Eq. (58). Moreover, note that the inversion of the symmetric matrix product indicated in Eq. (68) is unavoidable in the context of this non-linear frequency-dependent formulation.

7. Examples

7.1. Finite element analysis of a fixed-free bar

Consider a fixed-free, uniform bar subjected to a sudden application of constant axial force at the free end. The bar is discretized with six equally spaced elements. Fig. 1 shows the displacement response of the free extremity along time, considering one through four frequency terms, according to Eq. (10), and taking into account all $n = 6$ vibration modes in each solution, in order to demonstrate the improvement in accuracy. Target results is the analytical solution of the problem with 100 terms of the series. In all cases, time integration has been performed using Duhamel's integral.

7.2. Boundary element analysis of a fixed-free bar

Fig. 2 illustrates a fixed-free bar submitted to a periodic load at its free end. The circular frequency is 80% of the first natural frequency. The structure is analyzed as a two-dimensional (12×6 m^2) problem of potential with 36 linear boundary elements, as indicated. The normalized displacement response of node A is given in Fig. 3.

Cases with one, two and three frequency terms were considered, as labeled as HBEM_2, HBEM_4 and HBEM_6 in Fig. 3, respectively. Note how additional frequency terms dramatically improve the results. Notwithstanding, the authors think that the used mesh discretization should yield still better results. An implementation with higher-order boundary elements is being accomplished.

7.3. Numerical example: plane frame subjected to a periodic load

Fig. 4 illustrates a plane frame submitted to a horizontal periodic force. A target analysis was performed using a total of 24 beam elements with traditional, frequency-independent mass and stiffness matrices. Then, the analysis was repeated using only three beam elements for the frame, initially with frequency-

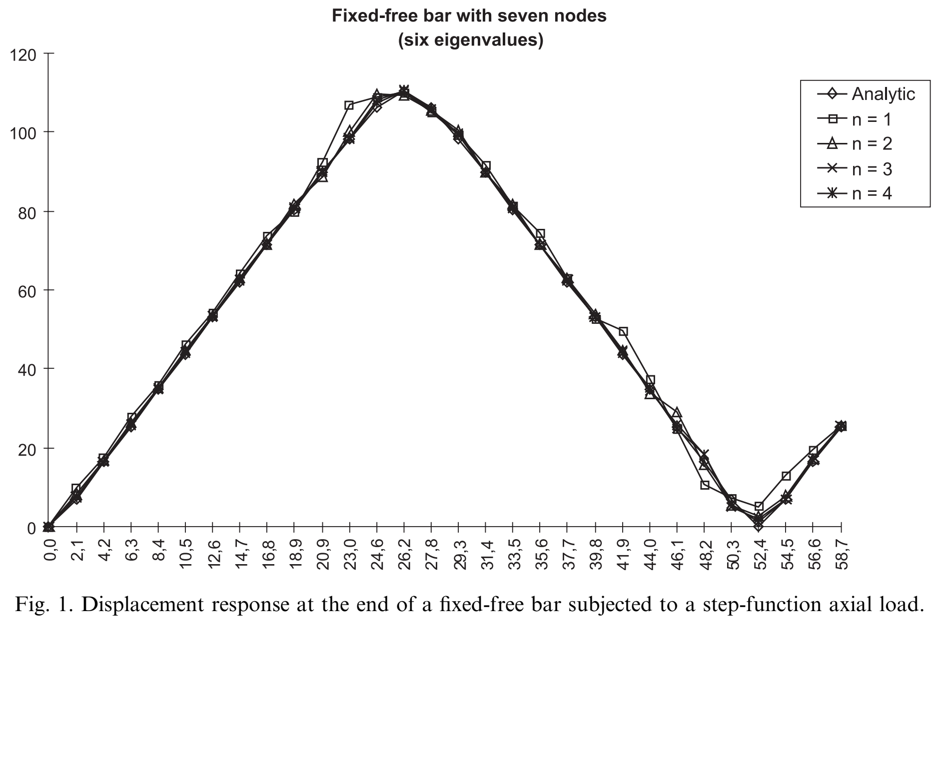
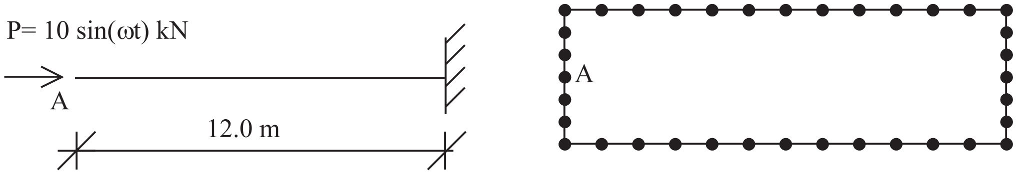


Fig. 1. Displacement response at the end of a fixed-free bar subjected to a step-function axial load.



$$E = 2.1 kN/m^2 \qquad \omega = 0.8\,\omega_1 = 0.09598\, rad/s \qquad Area = 6m^2 \qquad \rho = \frac{2500}{6} kg/m^3$$

Fig. 2. Fixed-free bar submitted to a periodic load at its free end. The structure is discretized with 36 linear boundary elements.

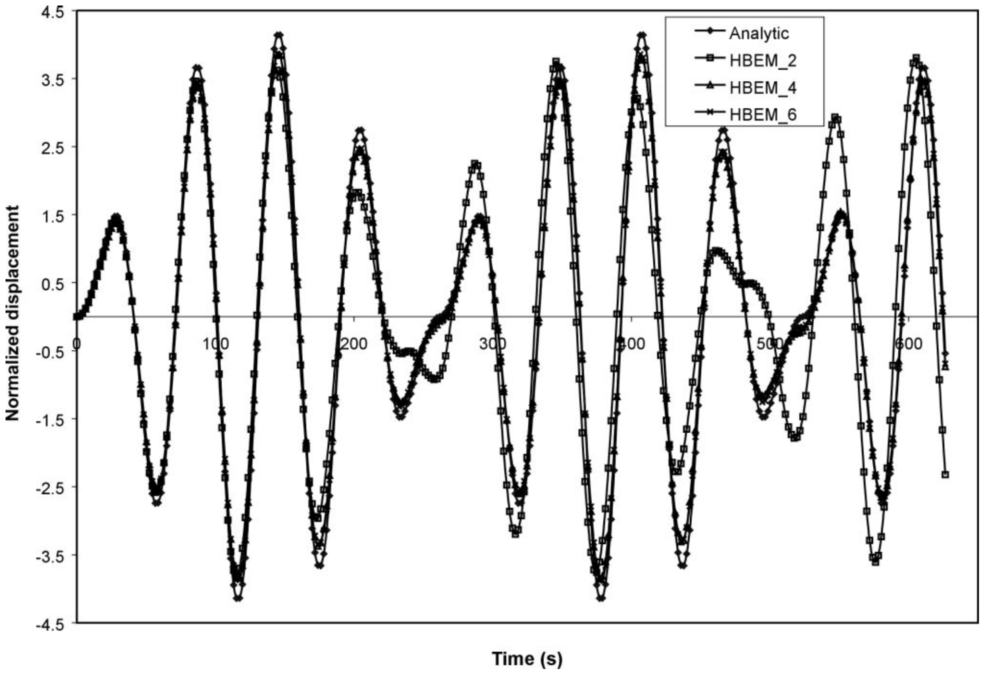


Fig. 3. Normalized displacement response of node A of the fixed-free bar in Fig. 2 as compared with the analytical solution.

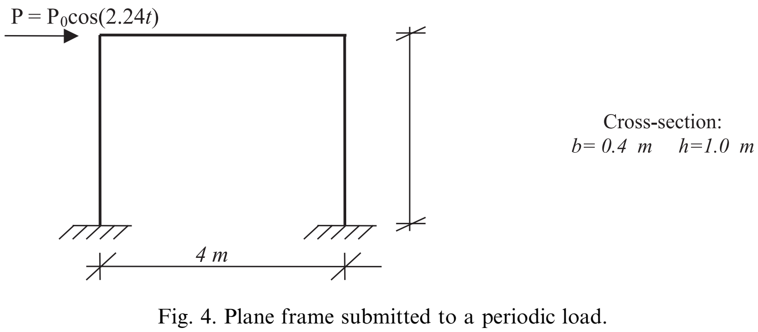


Fig. 4. Plane frame submitted to a periodic load.

Table 1
Natural vibration frequencies of the plane frame of Fig. 4

Frequencies	P24_2	P3_2	Error (%)	P3_4	Error (%)	P3_6	Error (%)	P3_8	Error (%)
1	2.8197	2.8330	0.467	2.8198	0.000	2.8196	0.006	2.8196	0.006
2	7.1035	8.2179	13.561	7.3454	3.293	7.1696	0.923	7.122	0.272
3	12.959	13.598	4.701	13.030	0.546	12.963	0.030	12.954	0.043
4	14.357	16.266	11.736	14.867	3.429	14.529	1.178	14.418	0.417
5	17.369	21.320	18.529	19.207	9.568	18.721	7.221	18.556	6.396
6	18.542	27.491	32.552	21.538	13.911	19.942	7.022	19.288	3.871

independent mass and stiffness matrices, and subsequently adding frequency-dependent terms, according to Eq. (49). The first six free-vibration frequencies of this plane frame are displayed in Table 1, as evaluated according to Eq. (51) for the different models. The models are characterized as Px_y, in which x is the total number of elements and y is the highest frequency power in the series expansion of the generalized mass matrix. Note the extreme accuracy of the frequency values for the model P3_8. The displacement response of the point of application of the periodic force is displayed in Fig. 5 for each one of the discretized models. The results with higher frequency terms, for a discretization with only 6 degrees of freedom, almost coincide with the target solution, which has been carried out using 69 degrees of freedom.

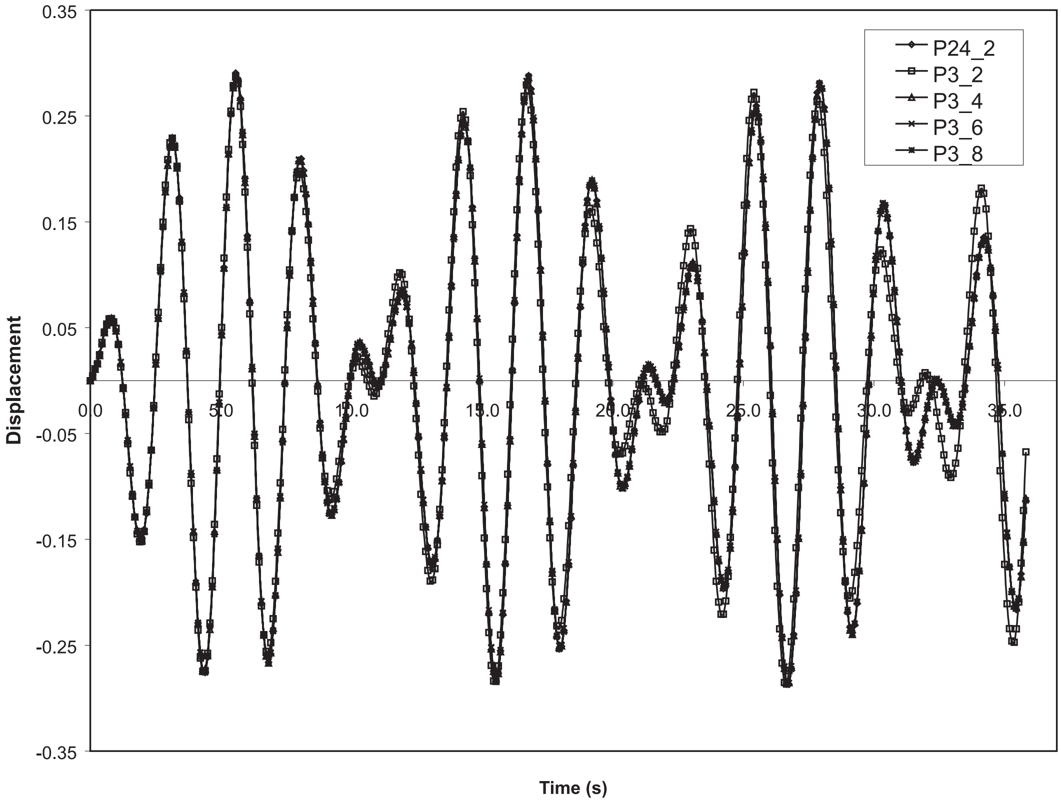


Fig. 5. Displacement response of the point of application of the periodic load of Fig. 4.

8. Conclusions

In Section 1, the authors illustrate how matrix equations with higher-order frequency terms may originate from conventional finite element displacement formulations, either in a direct implementation or as a result of dynamic condensation. In Section 2, it is shown that general frequency-dependent effective stiffness matrices are also obtained in the frame of the hybrid boundary element method, in a procedure that is applicable to Pian's hybrid finite elements, as a special case. In the core of the paper, a general set of coupled equations involving higher-order time derivatives is transformed into a set of uncoupled second-order differential equations of time, by means of an adequate mode superposition scheme. As a consequence, well-established time integration algorithms are directly applicable, for arbitrary loads as well as non-homogeneous boundary conditions. Although the theoretical developments are quite elaborate, it is straightforward to implement the method in an existing finite element package, provided that the non-linear eigenvalue problem expressed by Eq. (51) can be solved. In fact, the normalization procedure of Eq. (57) is rather uncomplicated, whereas the final second-order differential equation (63) looks traditional. Three academic examples are displayed for illustration of the method, with results compared for different numbers of frequency terms. The method is particularly advantageous in case of dynamic condensation, when high frequencies must be accounted for in terms of a small number of degrees of freedom.

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